# PHAS 1102: Physics of the Universe
# Problem Solving Tutorial 3 (Week 13)

You may assume the following:

$$
\begin{aligned}
G &= 6.67300 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}} \\
M_\odot &= 1.98892 \times 10^{30}\ \mathrm{kg} \\
\mathrm{pc} &= 3.08568025 \times 10^{16}\ \mathrm{m}
\end{aligned}
$$

and that

$$\int x \exp\left(\frac{-x}{a}\right)\,\mathrm{d}x = -a(a+x)\exp\left(\frac{-x}{a}\right)$$

$$\frac{mv^2}{r} = \frac{GM(r)m}{r^2}$$

A spiral galaxy is viewed from ‘above’. Its *average* luminosity per unit area, as a function of distance $r$ from the centre, is

$$L(r) = L_0 \exp\left(\frac{-r}{r_0}\right)$$

(smoothing out features like the spiral arms).

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Q1. Derive an expression for the total luminosity of the galaxy from the centre to some radius $R$. (You should start by writing down an integral, thinking about the area at each radius; then evaluate that integral.)

•The element of area of an annulus at radius $r$ is $2\pi r\mathrm{d}r$, so the product of the luminosity per unit area and the integrated area out to radius $R$ is

$$
\begin{aligned}
L(R) &= \int_0^R 2\pi r L_0 \exp\left(\frac{-r}{r_0}\right)\,\mathrm{d}r \\
&= 2\pi L_0 \int_0^R r\exp\left(\frac{-r}{r_0}\right)\,\mathrm{d}r \\
&= 2\pi L_0 \left[-r_0(r+r_0)\exp\left(\frac{-r}{r_0}\right)\right]_0^R \\
&= 2\pi L_0 r_0 \left[r_0 - (R+r_0)\exp\left(\frac{-R}{r_0}\right)\right]
\end{aligned}
$$

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Q2. For a particular galaxy, the central surface brightness is

$$L_0 = 150 L_\odot \text{ pc}^{-2}$$

and the scale length is

$$r_0 = 2 \text{ kpc}.$$

Compute the total luminosity (in units of $L_\odot$) within radius $R$ for $R = 0$, 2, 4, 6, 8, 10, 15, 20, and 25 kpc. Make a (rough) plot of your results.

•Just needs care with numerical work. I get these numbers (including velocities from the next question):

| $R$(kpc) | $L(R)/L_\odot$ | $V(R)$ (km/s) |
|---|---|---|
| 0.0 | 0.00000D+00 | 0.00000D+00 |
| 2.0 | 9.96166D+08 | 4.62855D+01 |
| 4.0 | 2.23931D+09 | 4.90705D+01 |
| 6.0 | 3.01914D+09 | 4.65221D+01 |
| 8.0 | 3.42467D+09 | 4.29100D+01 |
| 10. | 3.61750D+09 | 3.94456D+01 |
| 15. | 3.75219D+09 | 3.28013D+01 |
| 20. | 3.76803D+09 | 2.84666D+01 |
| 25. | 3.76972D+09 | 2.54670D+01 |

I also wrote a program to calculate the results at many intermediate radii; a plot is given later.

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Q3. Suppose, as a crude but reasonable approximation, that $M(R)/M_\odot$, the mass within radius $R$, equals $L(R)/L_\odot$, the luminosity within radius $R$, where both are measured in solar units. Calculate the orbital velocities of stars at the 9 radii used above.

• You might want to think about the way in which $M(R)$, the mass within radius $R$, is related to $L(R)$, the luminosity emitted within radius $R$. The question states that you should assume $L(R)/L_\odot \equiv M(R)/M_\odot$. The velocities then follow from

$$v = \sqrt{\frac{GM(R)}{R}}$$

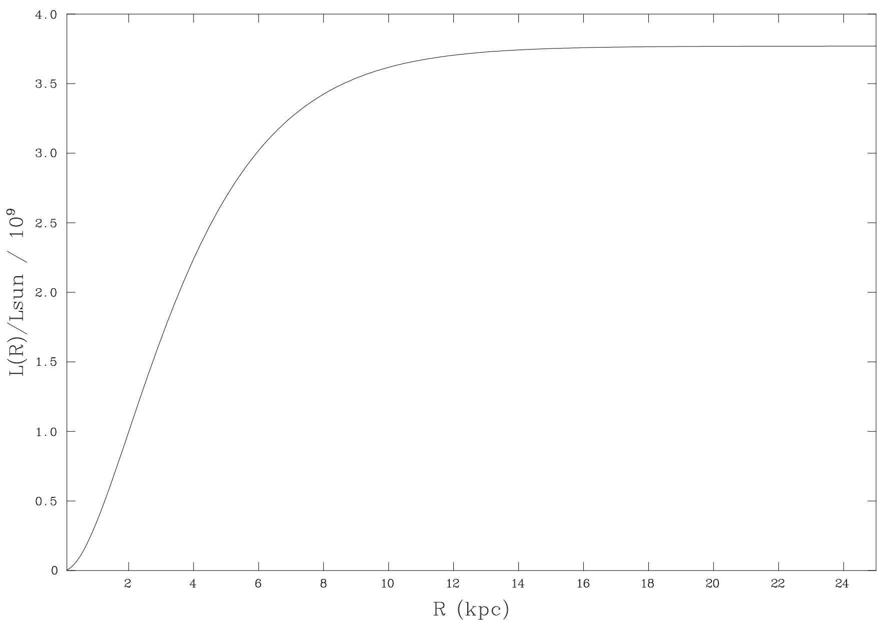
(i.e., from the balance between gravitional acceleration inwards and centrifugal acceleration outwards, as given in the preamble.)

The point of these three questions is to give students to get an idea of order-of-magnitude numbers, and to have them think about orbits in galaxies, on how observed rotation curves imply the existence of dark matter.

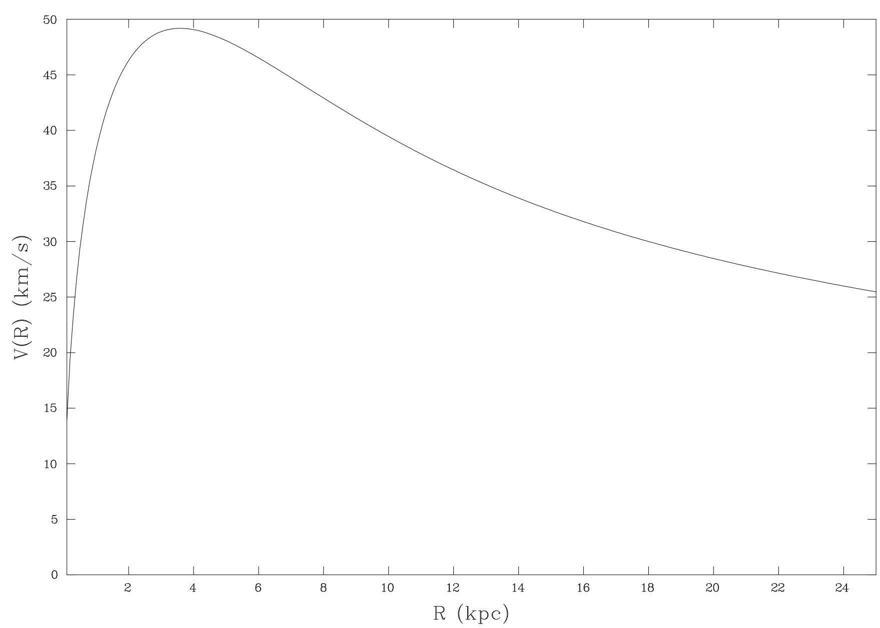
Q4. Suppose that the Sun's distance from the centre of the Galaxy is 8.0 kpc, and its orbital velocity about the centre is 220 km/s. Calculate the mass inside the solar orbit (in units of the solar mass).

•Mass:

$$\begin{aligned}
M(r) &= \frac{v^2 R}{G} \\
&= \frac{(220 \times 10^3)^2 \times 8000 \times 3.08568025 \times 10^{16}}{6.67300 \times 10^{-11}} \text{ kg} \\
&= 1.79 \times 10^{41} \text{ kg} \\
&= 9.0 \times 10^{11} M_\odot
\end{aligned}$$



4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
L(R)/Lsun / $10^9$
2
4
6
8
10
12
14
16
18
20
22
24
R (kpc)

50
45
40
35
30
25
20
15
10
5
0
V(R) (km/s)
2
4
6
8
10
12
14
16
18
20
22
24
R (kpc)