Problem Paper II:4 – Notes for Answers

Question 1 *The energy density of black-body radiation at temperature T is given by $(4\sigma/c)T^4$ where σ is the Stefan-Boltzmann constant and c is the speed of light. Calculate the energy density of the Cosmic Microwave Background ($T = 2.728$ K).*

A simple sum: $(4\sigma/c)T^4 = 4.19 \times 10^{-14}$ J m^{-3}

The typical energy of a photon in a black-body distribution of temperature T is roughly $3kT$, where k is Boltzmann's constant. What is the number density of CMB photons in the Universe today?

For 2.728 K, the average photon energy is $3kT = 1.13 \times 10^{-22}$ J;
so the photon number density is $(4.19 \times 10^{-14})/(1.13 \times 10^{-22})$ m^{-3}, i.e., 3.71×10^{8} m^{-3}

Question 2
In the Week 13 Problem-Solving Tutorial, you were invited to consider a spiral galaxy viewed from 'above'. You should've been able to show that the total luminosity of the disk, integrated from the centre out to some radius R, is given by

$$L(R) = 2\pi L_0 r_0 \left[r_0 - (R + r_0) \exp\left(\frac{-R}{r_0}\right) \right] \tag{1}$$

where L_0 is the central surface brightness and r_0 is the scale length for the particular galaxy. Here we will explore this same topic in more detail than was possible in the PST environment.

For a particular galaxy, we suppose the central surface brightness is

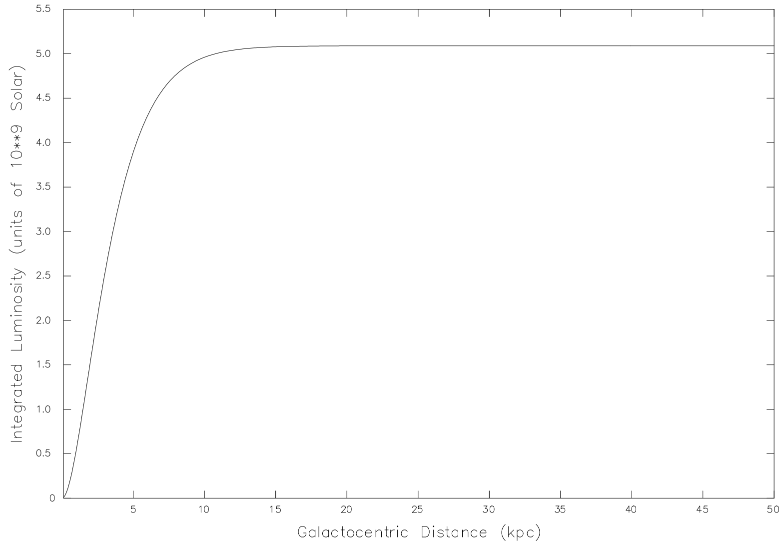
$$L_0 = 250 L_\odot \text{ pc}^{-2}$$

and the scale length is

$$r_0 = 1.8 \text{ kpc}.$$

By using a spreadsheet (or other technique of your choice), evaluate and plot the total luminosity (in units of $L_\odot$) for R =0–50 kpc at steps of 0.5 kpc.

This just requires evaluating equation 1. Most students will probably do this with Excel; I wrote a computer program. Here's what I get:



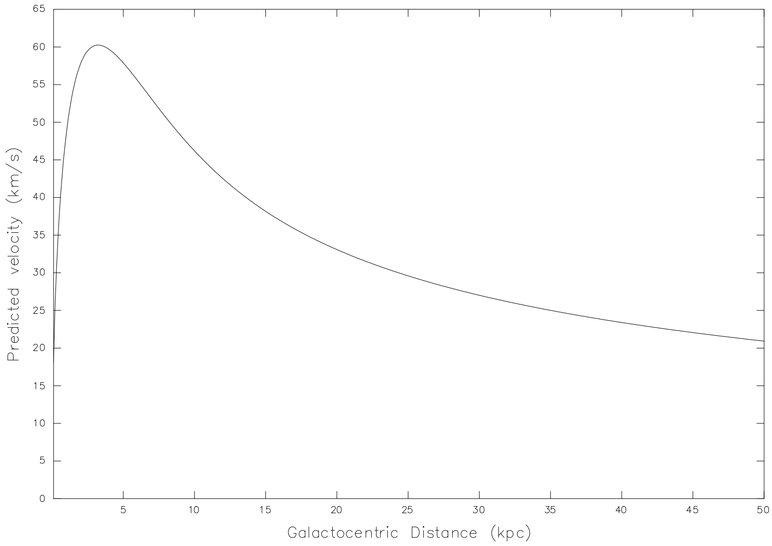
Suppose, as a crude but reasonable approximation, that $M(R)/M_{\odot}$, the mass within radius R, equals $L(R)/L_{\odot}$, the luminosity within radius R, where both are measured in solar units. Plot the predicted rotation curve (the orbital velocities of stars, in km/s, as a function of galactocentric distance, R, in kpc).

For example, if the galaxy emits $\sim 5 \times 10^9$ solar luminosities within a radius of 10 kpc, we suppose that this luminosity is generated by $\sim 5 \times 10^9$ solar masses of stars (i.e., by $\sim 5 \times 10^9$ stars if each star weighed one solar mass). So we can simply switch the luminosity data already calculated to mass data (in effect, change the y-axis label on the above diagram from 'Luminosity' to 'Mass'). If the mass contained within some radius R is $M(R)$, then we can estimate the orbital velocity by requiring that the centrifugal and gravitational forces balance (this is, in effect, the definition of a stable orbit); that is, for some star of mass m,

$$\frac{GM(R)m}{R^2} = \frac{mv^2(R)}{R}; \text{ i.e.,}$$

$$v(R) = \sqrt{\frac{GM(R)}{R}}.$$

Again, I wrote a computer program to compute $v(R)$, the orbital velocity at radius R. My results are shown overleaf; the important point about this figure is that, away from the centre, the *predicted* velocity falls off with increasing galactocentric distance. Observed velocities don't fall off in this way, so there must additional, unseen, mass ('dark matter') present.



Your answer should consist of two plots, plus a table listing $L(R)$, $M(R)$, and $v(R)$ for $R = 0, 10, 20, 30, 40,$ and 50 kpc.

[You may find it helpful to look at the notes on PST 3, available on the PHAS 1102 web page.]

The plots have already been given. Here's a selection of my numerical results (more numbers than are required from students):

R (kpc)	$L(R)/L_\odot$ $\equiv M(R)/M_\odot$	$v(R)$ (km s^{-1})
0.00000D+00	0.00000D+00	0.00000D+00
1.00000D+00	5.47081D+08	4.85087D+01
2.00000D+00	1.55245D+09	5.77813D+01
3.00000D+00	2.52602D+09	6.01799D+01
4.00000D+00	3.31224D+09	5.96794D+01
5.00000D+00	3.89394D+09	5.78767D+01
1.00000D+01	4.96040D+09	4.61904D+01
1.50000D+01	5.07796D+09	3.81586D+01
2.00000D+01	5.08846D+09	3.30805D+01
2.50000D+01	5.08931D+09	2.95906D+01
3.00000D+01	5.08937D+09	2.70125D+01
3.50000D+01	5.08938D+09	2.50088D+01
4.00000D+01	5.08938D+09	2.33936D+01
4.50000D+01	5.08938D+09	2.20557D+01
5.00000D+01	5.08938D+09	2.09238D+01