Problem Paper II:1 – Notes for Answers

Question 1

•*Derive a [Newtonian] expression for the Schwarzschild radius. . .by equating the kinetic energy required to eject a particle at the speed of light with that particle's potential energy.*

Just follow the instructions – equate the k.e. with the p.e.:

$$\left(\frac{1}{2}mv^2 =\right)\frac{1}{2}mc^2 = \frac{GMm}{R_S}$$

whence

$$R_S = \frac{2GM}{c^2}.$$

This is the right answer for a black hole of mass M (albeit somewhat fortuitously; you'd need a full relativistic treatment to *know* this is correct).

Question 2

•*Suppose, for the purposes of a rough calculation, that the Milky Way galaxy contains 10^{11} stars (and nothing else), each weighing one solar mass on average, and each composed entirely of hydrogen (not unreasonable for an order-of-magnitude estimate). Suppose further that the Milky Way is a typical galaxy, and that each cubic megaparsec of space contains 1 galaxy. On average, what volume is occupied by a single hydrogen atom? (Express your answer in units of cubic metres.)*

The total mass per Mpc^3 is $10^{11} \times \mathrm{M}_\odot = 1.99 \times 10^{41}$ kg

The mass of a hydrogen atom is 1.66×10^{-27} kg

The volume occupied by one hydrogen atom is therefore

$$(1.66 \times 10^{-27})/(1.99 \times 10^{41}) = 8.34 \times 10^{-69}\ \mathrm{Mpc}^3$$

or, in more sensible units,

$$8.34 \times 10^{-69} \times (3.086 \times 10^{22})^3\ \mathrm{m}^3 = 0.25\ \mathrm{m}^3.$$

(since 1 Mpc = 3.086×10^{22} m).

[*Very* roughly speaking, on average there's of order one atom per cubic metre in the local universe – compared with roughly one atom per cubic cm in interstellar gas in our Galaxy.]

Question 3

•*Another rough calculation: if the age of the Universe is 1.4×10^{10} years, estimate the volume of the observable universe. (Remember, this is intended to be a rough calculation, so the only extra information you should need to make this estimate is the speed of light. Express your answer in units of cubic megaparsecs.)*

If the age of the universe is t_0, we can see to a distance of something like $c \times t_0$:

$$1.4 \times 10^{10} \times (365.25 \times 24 \times 60 \times 60) \times 3.00 \times 10^{8} = 1.325 \times 10^{26} \text{ m},$$

or 4.33 Gpc; that is, a volume of 9.75×10^{78} m^3,
or, in more sensible units, 3.32×10^{11} Mpc^3.

[The actual size of the observable universe is not simply ct_0, but depends on the way in which the universe has expanded in the past. In general, the radius of the observable universe is somewhat larger than the simple estimate (by about a factor of 2–3), because the universe expanded while the light travelled across it. An alternative way of looking at this is to note that we see distant objects where they were, not where they are (and they're now further away).]

•*Suppose the European Standard Beach is 1km long, 10m across, and 1m deep; suppose also that the European Standard Sandgrain occupies 1 cubic mm. Which is the larger number: the number of stars in the observable universe, or the number of grains of sand on a beach? What about the number of galaxies compared to the number of sand grains?*

The volume of the Eurobeach is $1000 \times 10 \times 1 = 10^4$ m^3;
the volume occupied by the Standard Sandgrain is $1 \text{ mm}^3 = 10^{-9}$ m^3;
whence the number of grains in the beach is $10^4/10^{-9} = 10^{13}$.

Taking the volume of the observable universe as 3.32×10^{11} Mpc^3 (from above), then if there is one galaxy Mpc^{-3}, and 10^{11} stars in a galaxy (as stated in Q.2), then there are 3.32×10^{22} stars in the observable Universe.

[Note that, very roughly, there are as many galaxies in the observable universe as there are stars in the Galaxy.]

So there are more sandgrains on a beach than stars in a galaxy, and more sandgrains than galaxies – but many, many fewer sandgrains (on all the beaches of the world combined) than stars in the observable universe.

[You'll see that this is a very simple – indeed, almost trivial – calculation; and yet not so very long ago it was "announced" in a press release:
`http://news.bbc.co.uk/1/hi/sci/tech/3085885.stm`
Try this Google search:
`http://www.google.co.uk/search?q=simon+driver+stars+in+the+universe`
to see the astonishing amount of media coverage this "discovery" attracted!

As a matter of interest, even under the very best conditions, only something like 2000 stars can be seen with the naked eye at any one time; so the number of stars you can see without optical aid is comparable to the number of sand grains in just a handful of sand.]