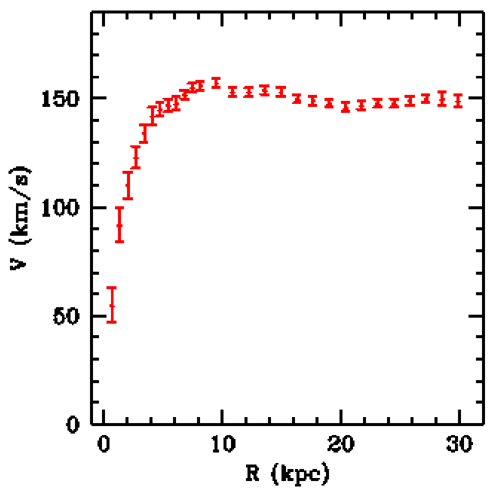
Physics of the Universe
In-Course Assessment, 2011 Dec 14

Assume the following values where necessary:

Speed of light	c	3.0×10^8 m s^{-1}
Gravitational constant	G	6.673×10^{-11} m^3 kg^{-1} s^{-2}
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
Stefan-Boltzmann constant	σ	5.670×10^{-8} W m^{-2} K^{-4}
hydrogen mass	$m_{\rm H}$	1.66×10^{-27} kg
Hubble constant	H_0	72 km s^{-1} Mpc^{-1}
solar mass	$M_\odot$	1.989×10^{30} kg
parsec	pc	3.0857×10^{16} m
year	yr	365.25d

All answers should be given in SI units unless indicated otherwise.

Question 1 – *Sketch the rotation curve of a spiral galaxy.*



The answer should be a figure something like the above (in particular, the velocity should be ∼constant away from the centre).
– 2 marks for the general form;
– 1 mark for each correctly labelled axis (“velocity” and “distance from centre”, or equivalent);
– 1 mark for each axis on which roughly correct numerical values are given (velocities of order 100–300 km/s, radii out to some like 20 kpc).
[6 marks maximum]

Question 2 – *The Virgo Cluster of galaxies is 16.5 Mpc distant. What do you expect its velocity of recession to be (in km s^{-1})?*

This is just a numerical exercise using the Hubble Law:

$$\begin{aligned} v = H_0 d &= 72 \text{ (km/s)/Mpc} \times 16.5 \text{ Mpc} \\ &= 1188 \text{ km/s} \end{aligned}$$

– 1 mark for Hubble's law (explicitly or implicitly),
– 1 mark for correct numerical result,
– 1 mark for right units.
[3 marks maximum]

Question 3 – *What is the Hubble time (in years)?*

Students have to know that the Hubble time is the inverse of the Hubble constant.

$$\begin{aligned} \tau = 1/H_0 &= \frac{\text{Mpc}}{72 \text{ km/s}} \\ &= \frac{3.0857 \times 10^{22} \text{ m}}{72000 \text{ m/s}} = 4.29 \times 10^{17} \text{ s} \\ &= \frac{4.29 \times 10^{17} \text{ s}}{60 \times 60 \times 24 \times 365.25 \text{ s/yr}} \\ &= 1.36 \times 10^{10} \text{ yr} \end{aligned}$$

– 2 marks for formulating answer correctly,
– 2 marks for numerical accuracy and units.
[4 marks maximum]

Question 4 – *Suppose that the Sun's distance from the centre of the Galaxy is 8.5 kpc, and its orbital velocity about the centre is 250 km/s. What is the mass of the Galaxy contained within the Sun's orbit? (Give your answer in kg, and in units of the solar mass.)*

This is an application of

$$\frac{GMm}{r^2} = \frac{mv^2}{r};$$

i.e.,

$$\begin{aligned} M &= \frac{v^2 r}{G} \\ &= \frac{(250 \times 10^3)^2 \times (8500 \times 3.0857 \times 10^{16})}{6.673 \times 10^{-11}} \ \frac{\text{m}^2 \text{ s}^{-2} \text{ m}}{\text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}} \\ &= 2.457 \times 10^{41} \text{ kg} \\ &= 1.235 \times 10^{11} \text{ M}_\odot \end{aligned}$$

– 2 marks for formulating answer correctly,
– 2 marks for numerical accuracy and units.
[4 marks maximum]

Question 5 – *A star is observed to orbit the black hole at the centre of our Galaxy. The orbit is circular, with an orbital period of 15.2 years and an orbital radius of 5.5 light-days.*
Determine the distance travelled by the star in one orbit (in metres), and hence its orbital velocity.
Use your results to estimate the mass of the Galactic-centre black hole, expressing your answer in units of the solar mass.

$$\begin{aligned} P_{\rm orb} &= 15.2\ \text{yr} = 4.797 \times 10^8\ \text{s}; \\ r &= 5.5\ \text{light-days} \\ &= (5.5 \times 24 \times 60 \times 60)\ \text{l-sec}\ \times\ 3 \times 10^8\ \text{m/s} \\ &= 1.4256 \times 10^{14}\ \text{m} \end{aligned}$$

Orbital distance is $2\pi r = 8.96 \times 10^{14}$ m.
– 2 marks
Velocity is $v = (2\pi r)/P_{\rm orb} = 1.87 \times 10^6$ m/s.
– 2 marks
Mass (as in Q4) is

$$\begin{aligned} M = (v^2 r)/G &= 7.45 \times 10^{36}\ \text{kg} \\ &= 3.75 \times 10^6\ \text{M}_\odot \end{aligned}$$

– 3 marks
[7 marks maximum]

Question 6 – *The energy density of black-body radiation at temperature T is given by $(4\sigma/c)T^4$ where σ is the Stefan-Boltzmann constant and c is the speed of light. Calculate the energy density of the Cosmic Microwave Background (CMB; assume $T = 2.73$ K), in units of J m^{-3}.*

$$\begin{aligned} \rho_{\rm CMB} &= \frac{4\ \times\ 5.670 \times 10^{-8}}{3.0 \times 10^8} \times 2.73^4\ \frac{\text{W m}^{-2}\ \text{K}^{-4}}{\text{m s}^{-1}} \text{K}^4 \\ &= 4.20 \times 10^{-14}\ \text{J m}^{-3} \end{aligned}$$

– 2 marks for numerical accuracy/units.

The typical energy of a photon in a black-body distribution of temperature T is roughly $3kT$, where k is Boltzmann's constant. What is the number density of CMB photons in the Universe today?

The average photon energy is

$$\begin{aligned} E_1 &= 3\ \times\ 1.381 \times 10^{-23}\ \times 2.73\ \text{J K}^{-1}\ \text{K}, \\ &= 1.13 \times 10^{-22}\ \text{J}; \end{aligned}$$

so the number density is

$$\rho_{\rm CMB}/E_1 = 3.71 \times 10^8\ \text{m}^{-3}.$$

– 1 mark for E_1, 1 for number density, 1 for overall numerical accuracy.

Suppose that the average number density of hydrogen atoms in the local universe is one per cubic metre. Express this as an energy density ($E = mc^2$). Compare (i) the number densities of hydrogen atoms and CMB photons, and (ii) the energy densities.

$$m_{\mathrm{H}}c^2 = 1.494 \times 10^{-10} \text{ J}$$

Thus while CMB photons outnumber hydrogen atoms by a factor 3.71×10^8, they contain a factor 1.32×10^{12} less energy.
– 3 marks
[8 marks total]

Question 7 – *The Friedmann equation describes the dynamical evolution of the Universe; it is given by*

$$\dot{R}^2 = \frac{8\pi G\rho(t)R^2(t)}{3} - kc^2 + \frac{\Lambda}{3}R^2(t)$$

where $R(t)$, $\rho(t)$ are the 'scale factor' and the mean density of the universe at time t; k is the curvature term; and Λ is the cosmological constant.

If $\Lambda = 0$, space has a flat geometry ($k = 0$) if the density equals the 'critical density', ρ_{c}.
Re-arrange the Friedmann equation into a form giving the critical density for a $k = 0$, $\Lambda = 0$ universe.
Thereby derive the present-day value for ρ_{c} (in kg m^{-3}).
How many hydrogen atoms per cubic metre are required to match this critical density?

(Recall that the Hubble constant $H_0 \equiv \dot{R}/R$.)

Rearranging as requested, we immediately obtain

$$\begin{aligned}\rho_{\mathrm{crit}} &= \frac{3}{8\pi G}\left(\frac{\dot{R}}{R}\right)^2 \\ &= \frac{3}{8\pi G}H_0^2 \\ &= \frac{3}{8\pi \times 6.673 \times 10^{-11}}\left(\frac{72 \times 10^3}{3.0857 \times 10^{22}}\right)^2 \\ &= 9.74 \times 10^{-27} \text{ kg m}^{-3} \\ &\simeq 6 \text{ H atoms m}^{-3}\end{aligned}$$

2 marks for ρ_{crit} formulation;
2 marks for correct evaluation in kg m^{-3};
1 mark for 6 H atoms.
[5 marks total]

Maximum mark for paper: 37